7. Geometry & Camera Model

Outline

- Basic Projective Geometry
- Camera Model

Homogeneous coordinates

Planar lines in Euclidean geometry: $ax + by + c = 0$

Multiple equations correspond to the same line

$$(ka)x + (kb)y + kc = 0, \forall k \neq 0$$

Homogeneous representation of planar lines: $(a,b,c)^\mathsf{T} \sim k(a,b,c)^\mathsf{T}$

Planar points in Euclidean geometry: $\mathrm{x} = (x, y)^\mathsf{T}$

Homogeneous representation of points

$$\mathrm{x} = (x, y, 1)^\mathsf{T} \qquad (x, y, 1)^\mathsf{T} \sim k(x, y, 1)^\mathsf{T}, \forall k \neq 0$$

Homogeneous coordinates $(x_1, x_2, x_3)^\mathsf{T}$ but only 2DOF

Points on lines:

x on l if and only if $\mathrm{x}^\mathsf{T} l = (x, y, 1)(a, b, c)^\mathsf{T} = ax + by + c = 0$

Points from lines and vice-versa

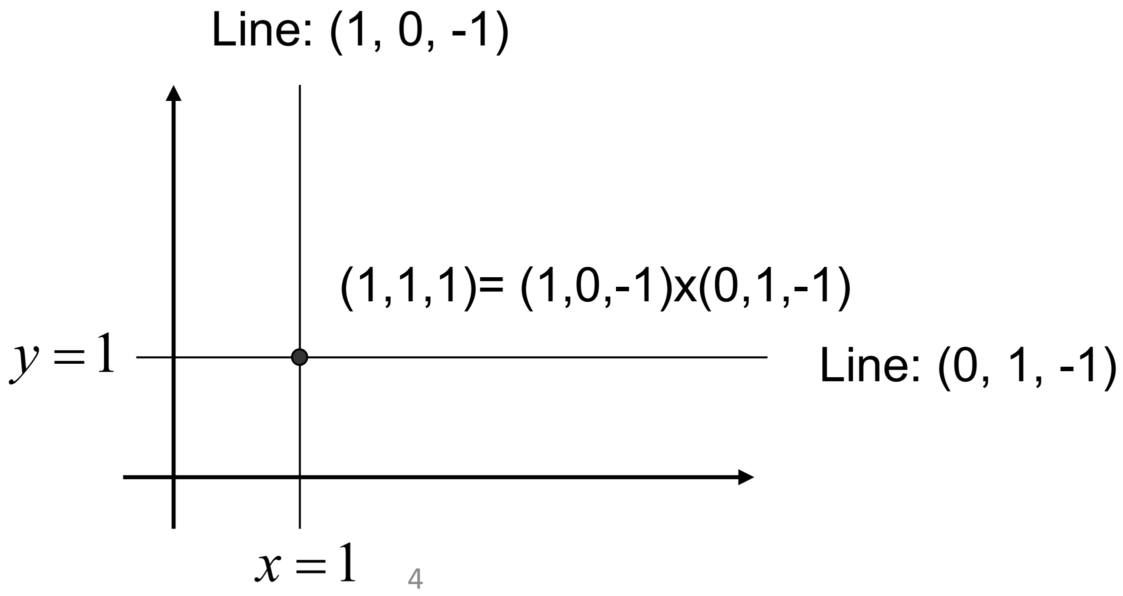
Intersections of lines

The intersection of two lines $\mathbf{l}$ and $\mathbf{l}'$ is $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$

Line joining two points

The line through two points $\mathbf{x}$ and $\mathbf{x}'$ is $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$

Example:



Ideal points and the line at infinity

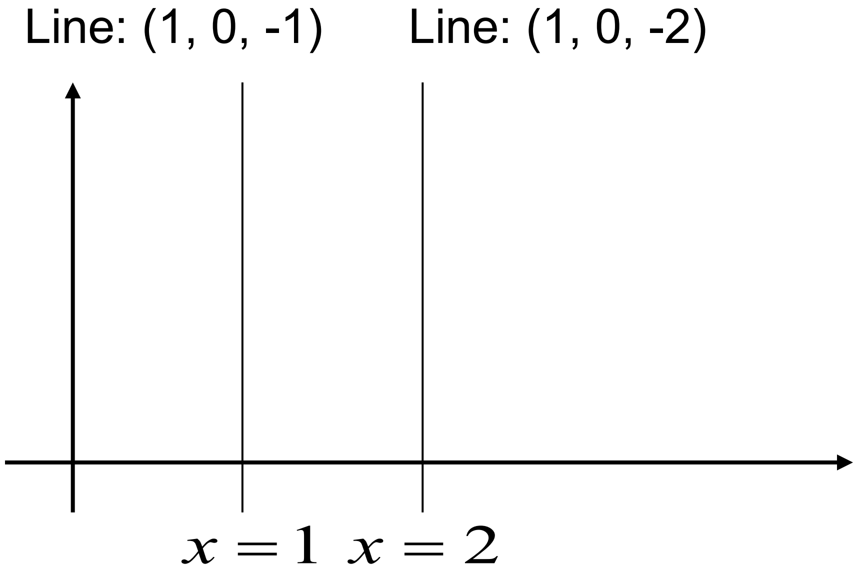
Intersections of parallel lines

$$\mathbf{l} = (a, b, c)^{\mathsf{T}} \text{ and } \mathbf{l}' = (a, b, c')^{\mathsf{T}} \qquad \mathbf{l} \times \mathbf{l}' = (b, -a, 0)^{\mathsf{T}}$$

Parallel lines intersect at ideal (imaginary) points $(x_1, x_2, 0)^{\mathsf{T}}$

All ideal points form a line, the line at infinity $\mathbf{l}_\infty = (0, 0, 1)^{\mathsf{T}}$

Example



$\mathbf{P}^2 = \mathbf{R}^2 \cup \mathbf{l}_\infty$ Note that in P^2 there is no distinction between ideal points and others

Conics

Curve described by 2nd-degree equation in the plane

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

or *homogenized* $\quad x \mapsto x_1/x_3,\ y \mapsto x_2/x_3$

$$a{x_1}^2 + bx_1x_2 + c{x_2}^2 + dx_1x_3 + ex_2x_3 + fx_3^2 = 0$$

or in matrix form

$$\mathbf{x}^{\mathsf{T}}\mathbf{C}\,\mathbf{x} = 0 \quad \text{with} \quad \mathbf{C} = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$$

5DOF: $\{a : b : c : d : e : f\}$

Five points define a conic

For each point the conic passes through

$$ax_i^2 + bx_i y_i + cy_i^2 + dx_i + ey_i + f = 0$$

or

$$(x_i^2, x_i y_i, y_i^2, x_i, y_i, 1)\mathbf{c} = 0 \qquad \mathbf{c} = (a, b, c, d, e, f)^{\mathrm{T}}$$

stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4 y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5 y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} \mathbf{c} = 0$$

Questions?

Projective transformations

Definition:

A *projectivity* is an invertible mapping h from P^2 to itself such that three points x_1,x_2,x_3 lie on the same line if and only if $h(x_1),h(x_2),h(x_3)$ do.

Theorem:

A mapping $h:P^2 \rightarrow P^2$ is a projectivity if and only if there exist a non-singular 3x3 matrix H such that for any point in P^2 reprented by a vector x it is true that $h(x)=Hx$

Definition: Projective transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad \mathbf{x}' = \mathbf{H}\,\mathbf{x}$$

8DOF

projectivity=collineation=projective transformation=homography

A hierarchy of transformations

Projective linear group

Affine group (last row (0,0,1))

Euclidean group (upper left 2x2 orthogonal)

Oriented Euclidean group (upper left 2x2 det 1)

Alternative, characterize transformation in terms of elements or quantities that are preserved or *invariant*

e.g. Euclidean transformations leave distances unchanged



Decomposition of projective transformations

$$\mathbf{H} = \mathbf{H}_S \mathbf{H}_A \mathbf{H}_P = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{K} & 0 \\ 0^\mathsf{T} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & 0 \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^\mathsf{T} & v \end{bmatrix}$$

$\mathbf{A} = s\mathbf{R}\mathbf{K} + \mathrm{t}\mathrm{v}^\mathsf{T}$ $\mathbf{K}$ upper-triangular, $\det \mathbf{K} = 1$

decomposition unique (if chosen s>0)

Similar unique de-composition: $\mathbf{H} = \mathbf{H}_P \mathbf{H}_A \mathbf{H}_S$

Example:

$$\mathbf{H} = \begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 2\cos 45^\circ & -2\sin 45^\circ & 1.0 \\ 2\sin 45^\circ & 2\cos 45^\circ & 2.0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

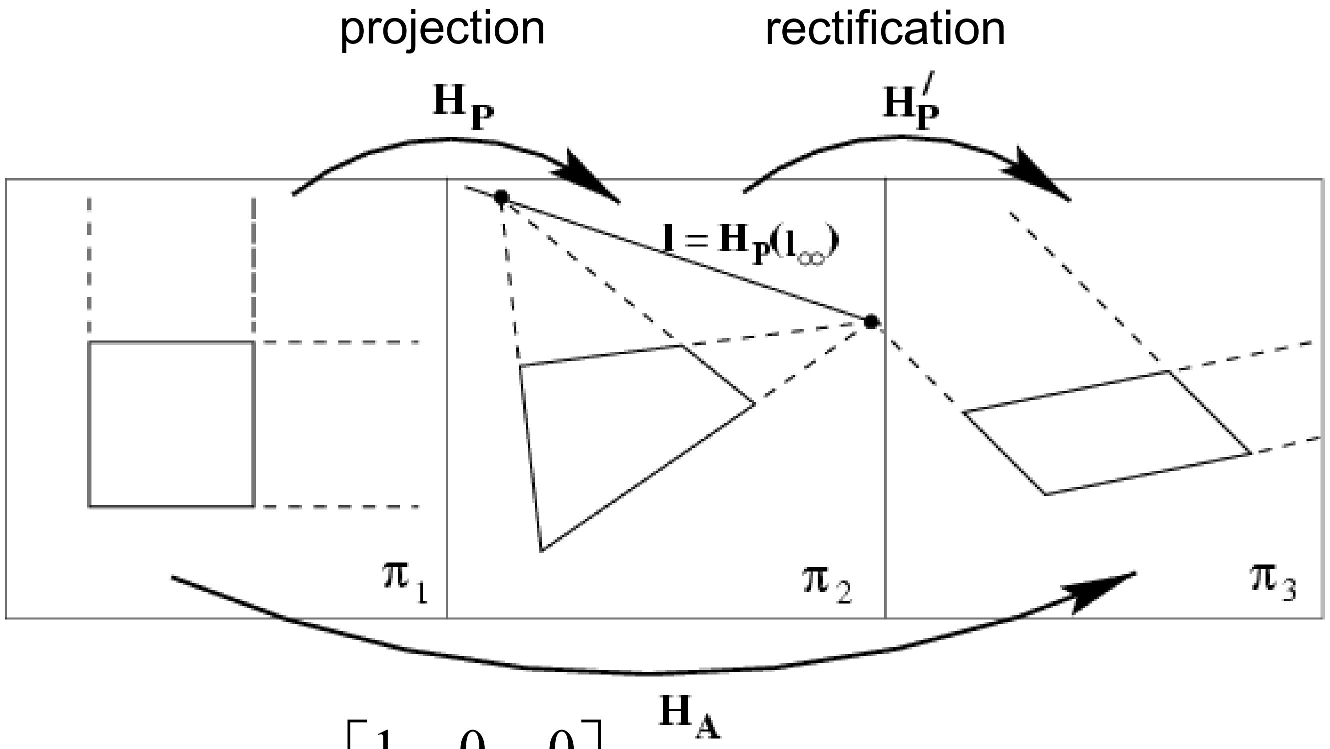
The line at infinity

$$\mathbf{l}'_\infty = \mathbf{H}_A^{-T}\, \mathbf{l}_\infty = \begin{bmatrix} \mathbf{A}^{-T} & 0 \\ -(\mathbf{A}^{-1}\mathbf{t})^{T} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \mathbf{l}_\infty$$

The line at infinity l_∞ is a fixed line under a projective transformation H if and only if H is an affine

Note: not fixed pointwise

Affine properties from images

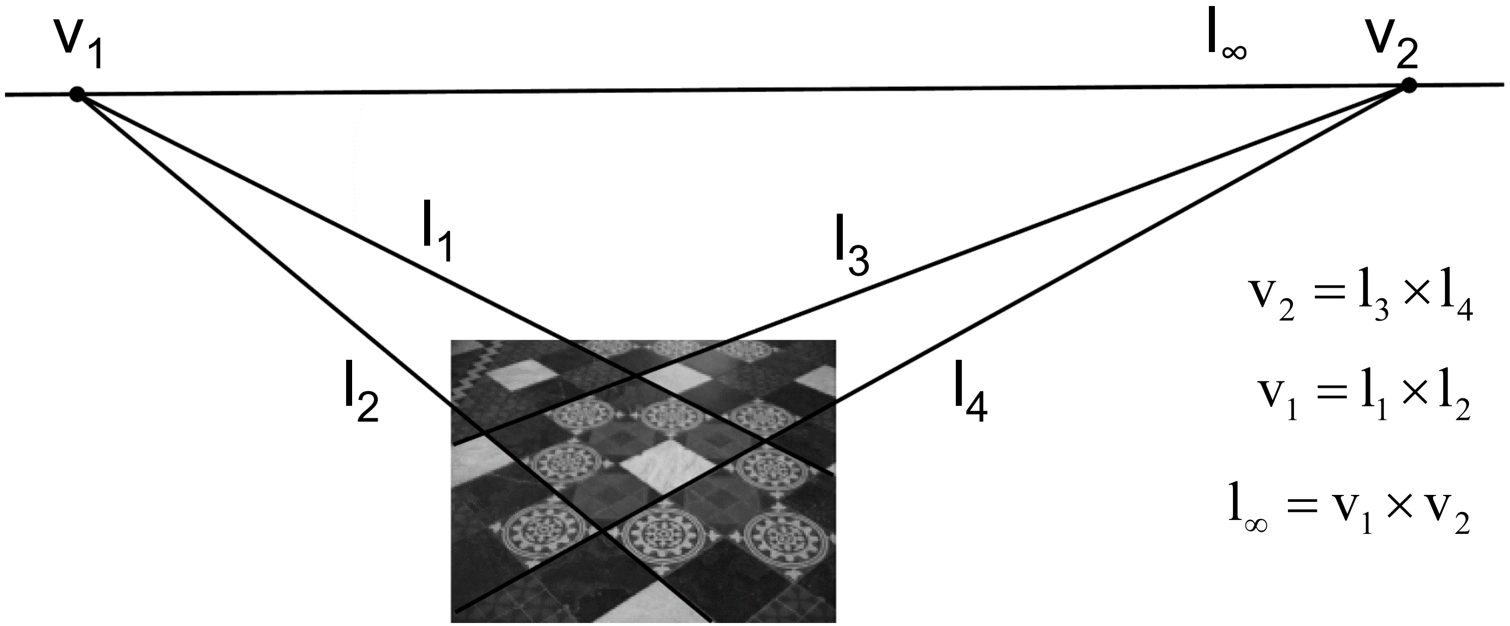


$$\mathbf{H'}_P = \mathbf{H}_A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix} \qquad \mathbf{l}_\infty = [l_1 \quad l_2 \quad l_3]^\mathsf{T}, l_3 \neq 0$$

H'_p maps the l_∞ back to its canonical position (0,0,1)

It can be easily verified by checking $H'_p{}^{-T} l_\infty = [0,0,1]'$

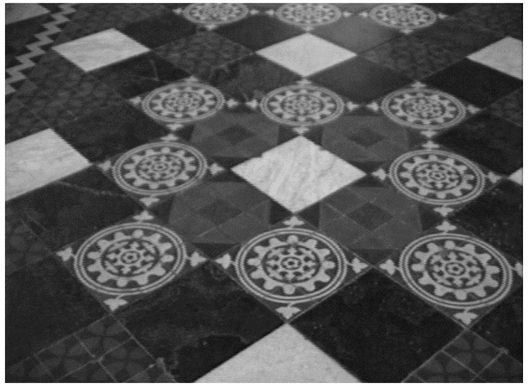
Affine rectification



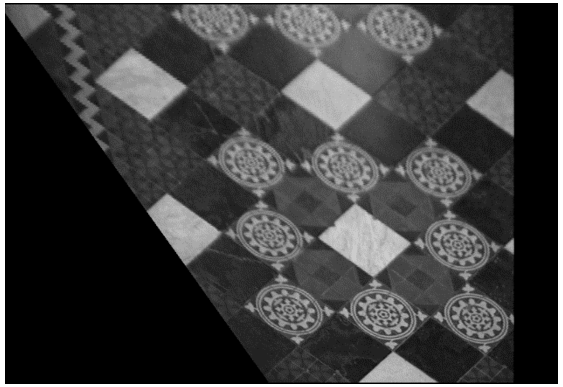
$$v_2 = l_3 \times l_4$$

$$v_1 = l_1 \times l_2$$

$$l_\infty = v_1 \times v_2$$



$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$



Questions?

3D points

Homogenous coordinate of 3D points

$$(X,Y,Z)^{\mathsf{T}} \text{ in } R^3$$

$$\mathrm{X} = (X_1, X_2, X_3, X_4)^{\mathsf{T}} \text{ in } P^3$$

$$\mathrm{X} = \left(\frac{X_1}{X_4}, \frac{X_2}{X_4}, \frac{X_3}{X_4}, 1\right)^{\mathsf{T}} = (X, Y, Z, 1)^{\mathsf{T}} \qquad (X_4 \neq 0)$$

Projective transformation in 3D

$$\mathrm{X}' = \mathbf{H}\,\mathrm{X} \qquad \text{(4x4-1=15 dof)}$$

Quadrics

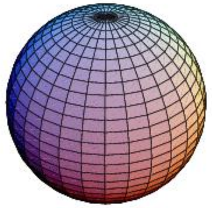
$$X^T Q X = 0$$ (Q : 4x4 symmetric matrix)

$$Q = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \bullet \end{bmatrix}$$

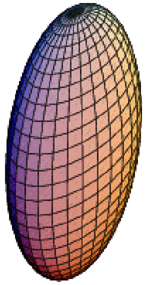
1. 9 d.o.f.
2. in general 9 points define a quadric
3. det Q=0 ↔ degenerate quadric
4. (plane ∩ quadric)=conic
5. transformation $Q' = H^{-T} Q H^{-1}$

Quadric classification

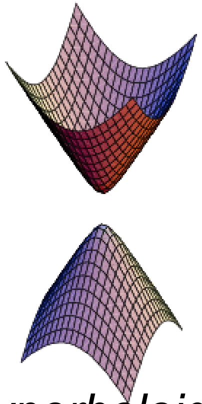
Projectively equivalent to *sphere:*



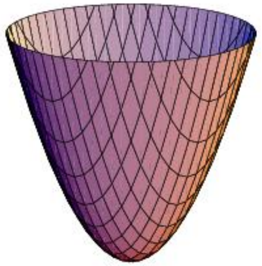
sphere



ellipsoid



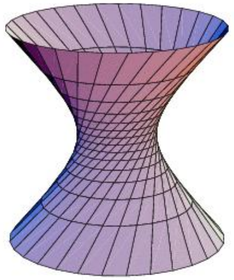
hyperboloid of two sheets

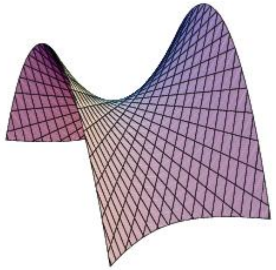


paraboloid

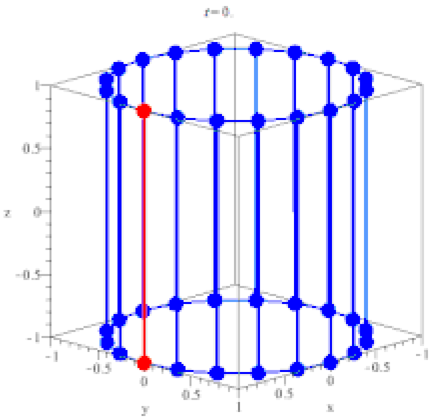
Ruled quadrics:

hyperboloids of one sheet

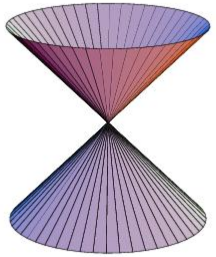




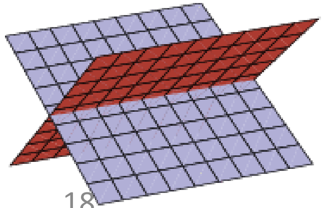
Hyperbolic paraboloid



Degenerate ruled quadrics:



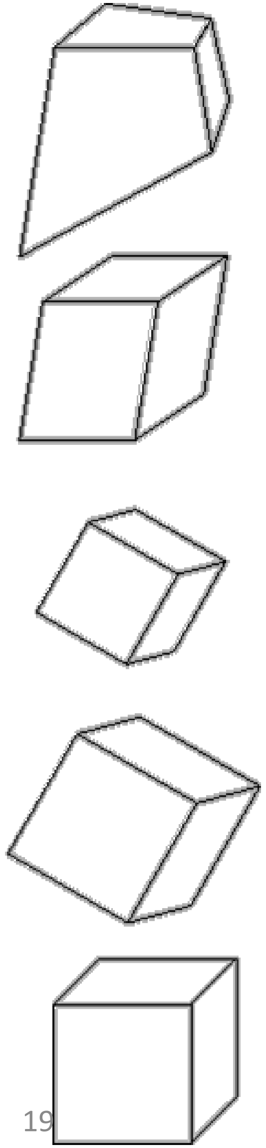
cone



two planes

Hierarchy of transformations

Projective 15dof	$\begin{bmatrix} \mathrm{A} & \mathrm{t} \\ \mathrm{v}^{\mathrm{T}} & v \end{bmatrix}$	Intersection and tangency
Affine 12dof	$\begin{bmatrix} \mathrm{A} & \mathrm{t} \\ 0^{\mathrm{T}} & 1 \end{bmatrix}$	Parallellism of planes, Volume ratios, centroids, The plane at infinity π_∞
Similarity 7dof	$\begin{bmatrix} s\,\mathrm{R} & \mathrm{t} \\ 0^{\mathrm{T}} & 1 \end{bmatrix}$	The absolute conic Ω_∞
Euclidean 6dof	$\begin{bmatrix} \mathrm{R} & \mathrm{t} \\ 0^{\mathrm{T}} & 1 \end{bmatrix}$	Volume



The plane at infinity

$$\pi'_\infty = \mathbf{H}_A^{-\mathsf{T}}\pi_\infty = \begin{bmatrix} \mathbf{A}^{-\mathsf{T}} & 0 \\ -\mathbf{A}\,\mathrm{t} & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \pi_\infty$$

The plane at infinity π_∞ is a fixed plane under a projective transformation H iff H is an affinity

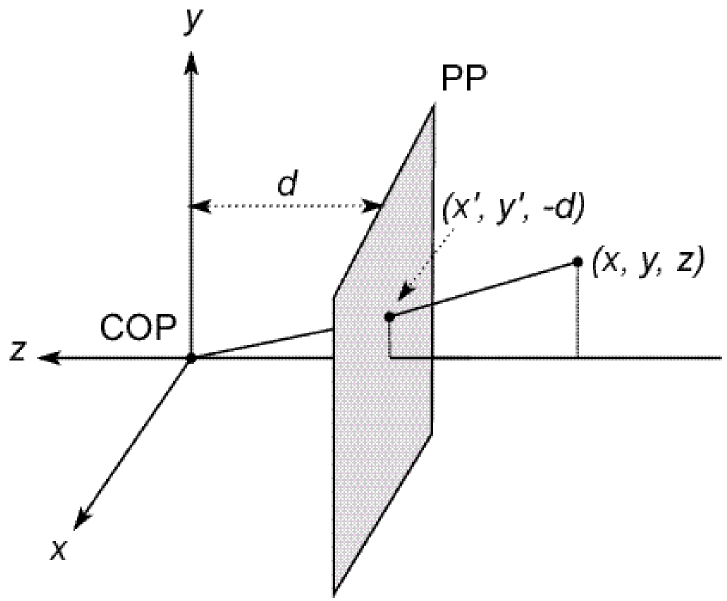
1. canonical position $\pi_\infty = (0,0,0,1)^\mathsf{T}$
2. contains directions $\mathrm{D} = (X_1, X_2, X_3, 0)^\mathsf{T}$
3. two planes are parallel $\Leftrightarrow$ line of intersection in π_∞
4. line // line (or plane) $\Leftrightarrow$ point of intersection in π_∞

Questions?

Outline

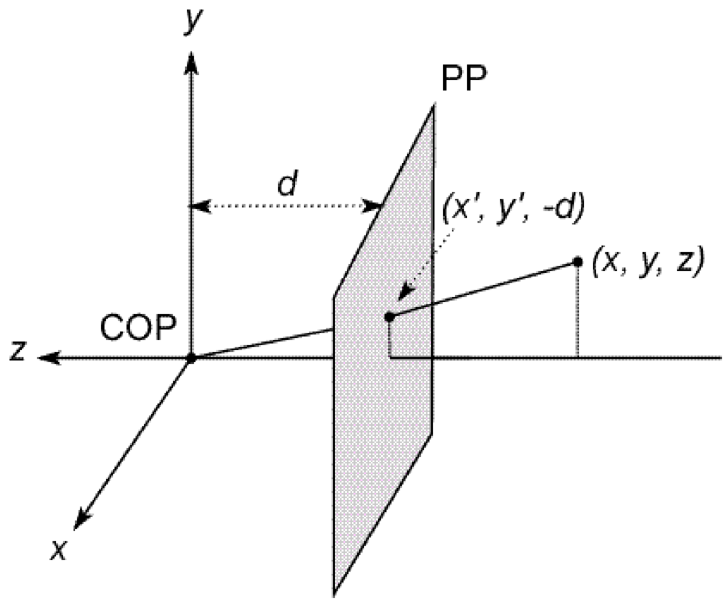
- Basic Projective Geometry
- Camera Model

Modeling projection



- The coordinate system
  - We will use the pinhole model as an approximation
  - Put the optical center (**C**enter **O**f **P**rojection) at the origin
  - Put the image plane (**P**rojection **P**lane) *in front* of the COP
    - Why?
  - The camera looks down the *negative* z axis
    - we need this if we want right-handed-coordinates

Modeling projection



- Projection equations
  - Compute intersection with PP of ray from (x,y,z) to COP
  - Derived using similar triangles

$$(x, y, z) \rightarrow (-d\frac{x}{z}, \ -d\frac{y}{z}, \ -d)$$

  - We get the projection by throwing out the last coordinate:

$$(x, y, z) \rightarrow (-d\frac{x}{z}, -d\frac{y}{z})$$

Modeling projection

- Is this a linear transformation?
  - no—division by z is nonlinear

Homogeneous coordinates to the rescue!

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

homogeneous image coordinates

$$(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

homogeneous scene coordinates

Converting *from* homogeneous coordinates

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

Perspective projection

Projection is a matrix multiply using homogeneous coordinates:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ -z/d \end{bmatrix} \Rightarrow (-d\frac{x}{z}, \quad -d\frac{y}{z})$$

divide by third coordinate

This is known as perspective projection

- The matrix is the projection matrix

- (Can also represent as a 4x4 matrix – OpenGL does something like this)

Perspective projection

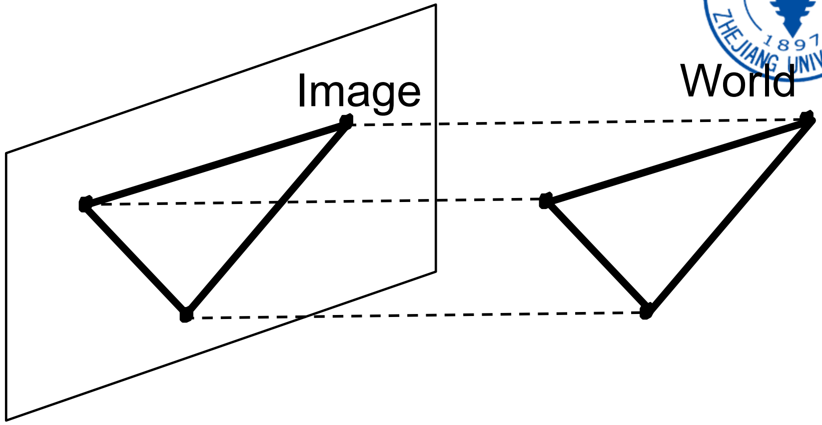
- How does scaling the projection matrix change the transformation?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/d & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ -z/d \end{bmatrix} \Rightarrow (-d\frac{x}{z}, \quad -d\frac{y}{z})$$

$$\begin{bmatrix} -d & 0 & 0 & 0 \\ 0 & -d & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} -dx \\ -dy \\ z \end{bmatrix} \Rightarrow (-d\frac{x}{z}, \quad -d\frac{y}{z})$$

Orthographic projection

- Special case of perspective projection
  - Distance from the COP to the PP is infinite
  - Good approximation for telephoto optics
  - Also called "parallel projection": (x, y, z) → (x, y)
  - What's the projection matrix?



$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \Rightarrow (x, y)$$

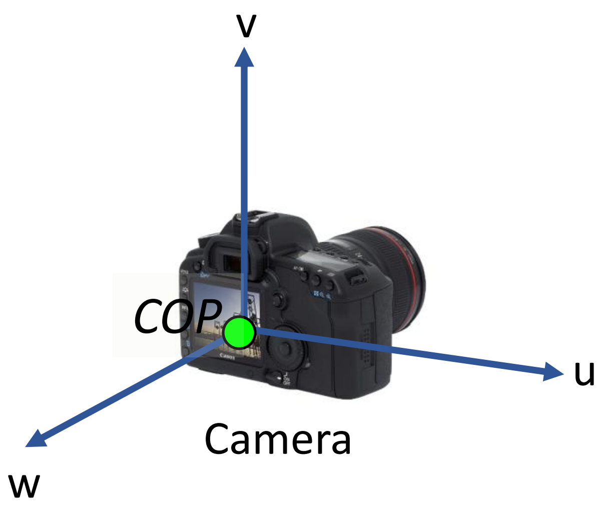
Questions?

Camera parameters

- How many numbers do we need to describe a camera?

- We need to describe its *pose* in the world
- We need to describe its internal parameters

A Tale of Two Coordinate Systems



Camera

"The World"

Two important coordinate systems:

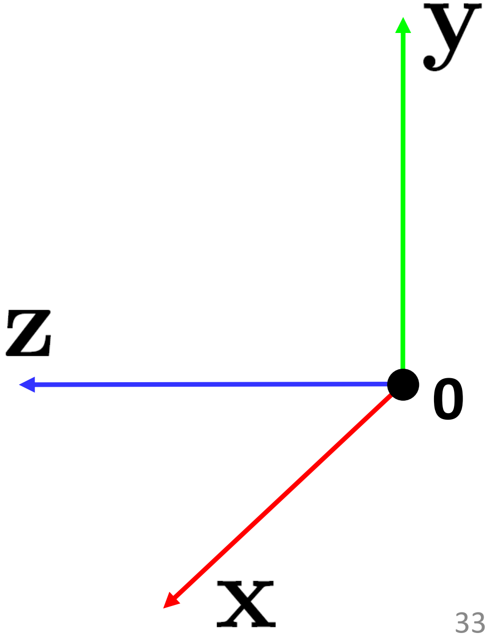
1. *World* coordinate system
2. *Camera* coordinate system

Camera parameters

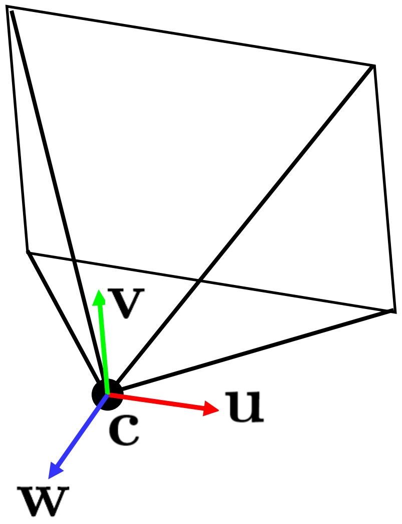
- To project a point (*x*,*y*,*z*) in *world* coordinates into a camera
- First transform (*x*,*y*,*z*) into *camera* coordinates
- Need to know
  - Camera position (in world coordinates)
  - Camera orientation (in world coordinates)
- Then project into the image plane
  - Need to know camera *intrinsics*
  - We mostly saw this operation last time
- These can all be described with matrices

Extrinsics

- How do we get the camera to "canonical form"?
  - (Center of projection at the origin, x-axis points right, y-axis points up, z-axis points backwards)



Step 1: Translate by -c



Extrinsics

- How do we get the camera to “canonical form”?
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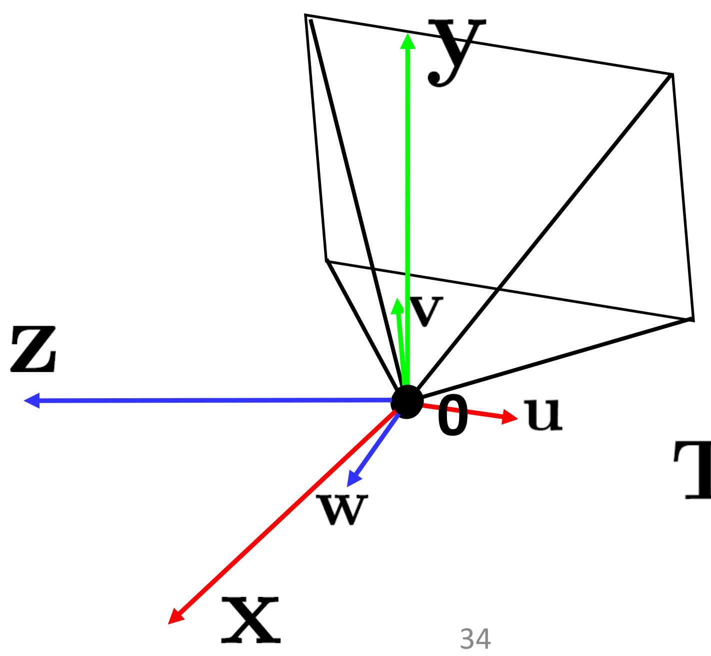


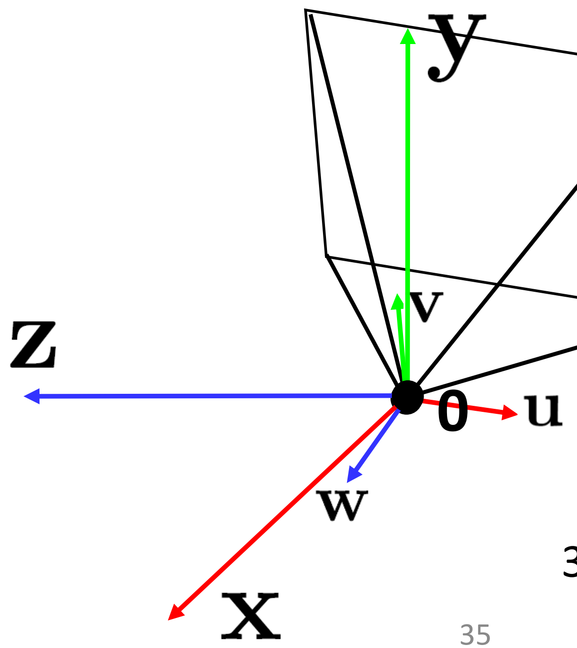
Step 1: Translate by -c

How do we represent translation as a matrix multiplication?

$$\mathbf{T} = \begin{bmatrix} \mathbf{I}_{3\times 3} & -\mathbf{c} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

Extrinsics

- How do we get the camera to "canonical form"?
  - (Center of projection at the origin, x-axis points right, y-axis points up, z-axis points backwards)



Step 1: Translate by -c

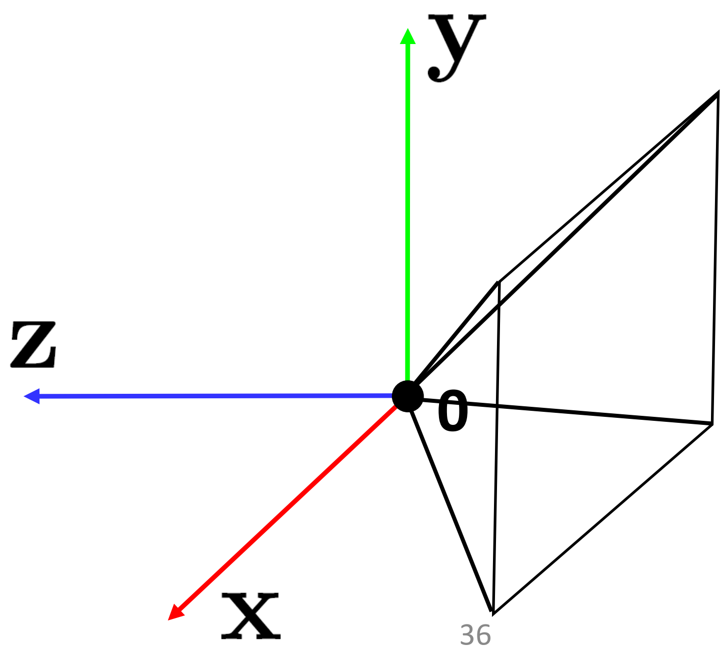
Step 2: Rotate by R

$$\mathbf{R} = \begin{bmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{bmatrix}$$

3x3 rotation matrix

Extrinsics

- How do we get the camera to "canonical form"?
  - (Center of projection at the origin, x-axis points right, y-axis points up, z-axis points backwards)



Step 1: Translate by -c

Step 2: Rotate by R

$$\mathbf{R} = \begin{bmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{bmatrix}$$

Perspective projection

$$\underbrace{\begin{bmatrix} -f & 0 & 0 \\ 0 & -f & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{K}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\mathbf{K}$ (intrinsics) (converts from 3D rays in camera coordinate system to pixel coordinates)

in general, $\mathbf{K} = \begin{bmatrix} -f & s & c_x \\ 0 & -\alpha f & c_y \\ 0 & 0 & 1 \end{bmatrix}$ (upper triangular matrix)

α : aspect ratio (1 unless pixels are not square)

s : skew (0 unless pixels are shaped like rhombi/parallelograms)

(c_x, c_y) : principal point ((0,0) unless optical axis doesn't intersect projection plane at origin)

Focal length

- Can think of as "zoom"

24mm

50mm

200mm

800mm

- Related to *field of view*

Projection matrix

$$\mathbf{\Pi} = \underset{\text{intrinsics}}{\mathbf{K}} \underbrace{\underset{\text{projection}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}} \underset{\text{rotation}}{\begin{bmatrix} \mathbf{R} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix}} \underset{\text{translation}}{\begin{bmatrix} \mathbf{I}_{3\times 3} & -\mathbf{c} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix}}}_{\left[\ \mathbf{R}\ \middle|\ \underbrace{-\mathbf{Rc}}_{\text{(t in book's notation)}}\ \right]}$$

$$\mathbf{\Pi} = \mathbf{K} \left[\ \mathbf{R}\ \middle|\ -\mathbf{Rc}\ \right]$$

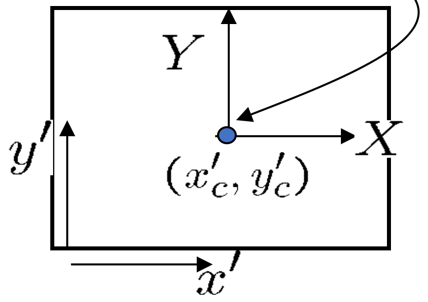
Camera parameters

A camera is described by several parameters

- Translation c of the optical center from the origin of world coords
- Rotation R of the image plane
- focal length f, principle point (x'_c, y'_c), skew, etc
- blue parameters are called "extrinsics," red are "intrinsics"

Projection equation

$$\mathbf{x} = \begin{bmatrix} sx \\ sy \\ s \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \mathbf{\Pi X}$$



- The projection matrix models the cumulative effect of all parameters
- Useful to decompose into a series of operations

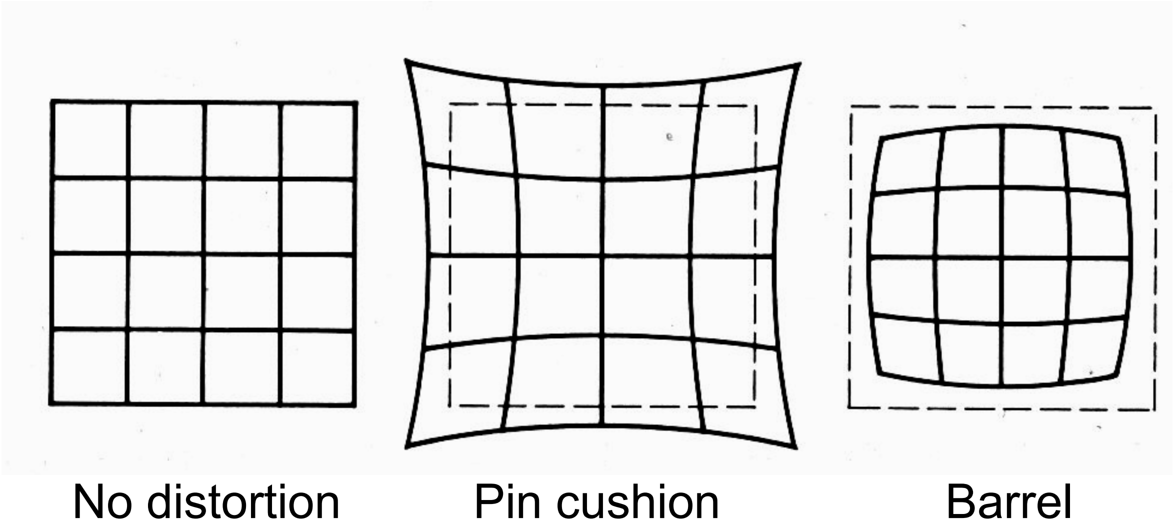
identity matrix

$$\mathbf{\Pi} = \begin{bmatrix} -fs_x & 0 & x'_c \\ 0 & -fs_y & y'_c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{R}_{3x3} & \mathbf{0}_{3x1} \\ \mathbf{0}_{1x3} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I}_{3x3} & \mathbf{T}_{3x1} \\ \mathbf{0}_{1x3} & 1 \end{bmatrix}$$

intrinsics | projection | rotation | translation

Questions?

Distortion

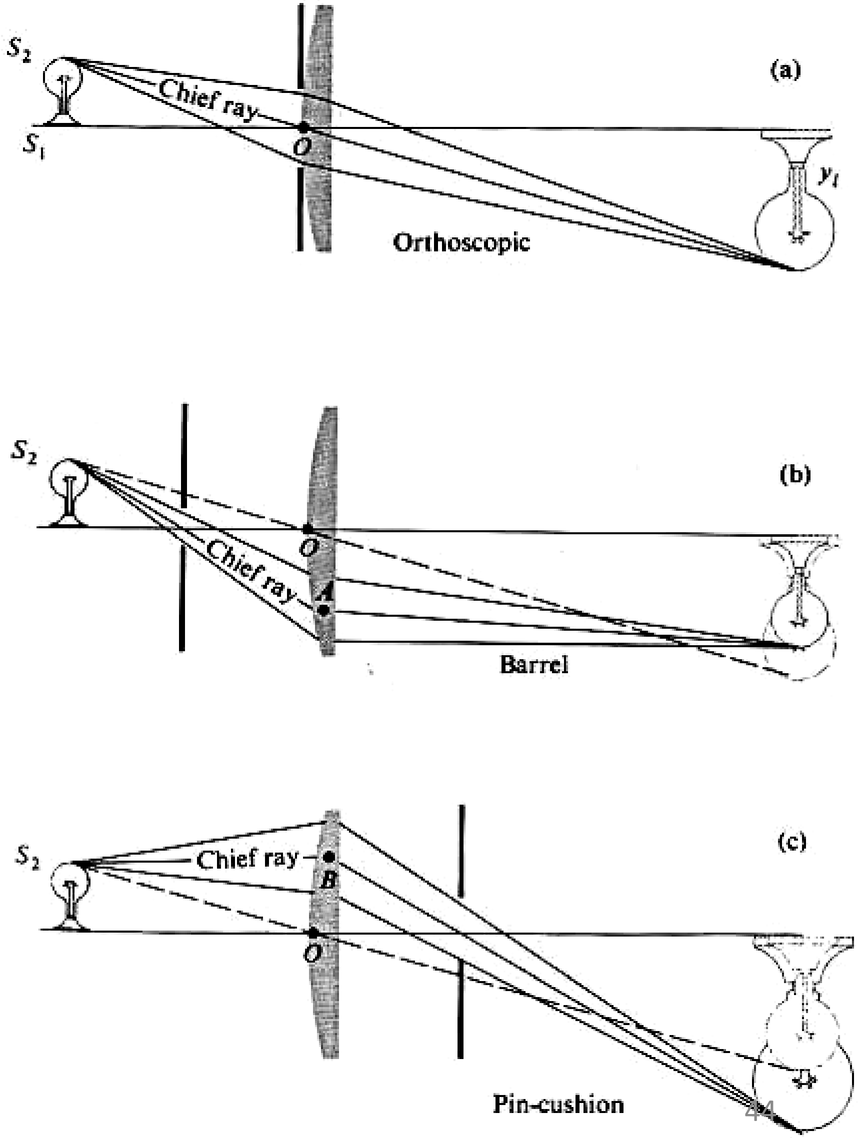


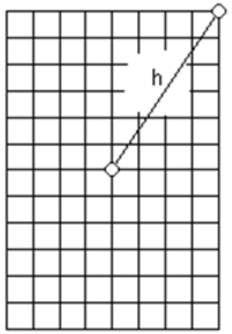
- Radial distortion of the image
  - Caused by imperfect lenses
  - Deviations are most noticeable for rays that pass through the edge of the lens

Correcting radial distortion

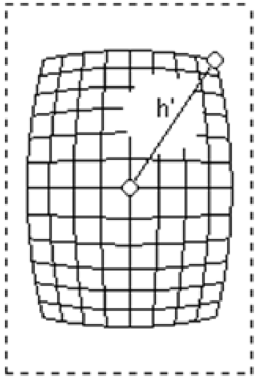
from Helmut Dersch

Distortion

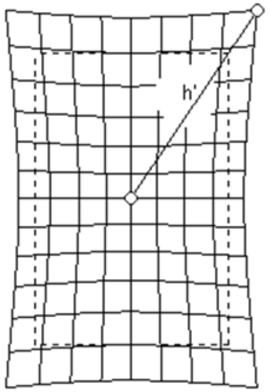




orthoscopic



barrel



pincushion

Modeling distortion

- To model lens distortion
  - Use the following operations instead of standard projection matrix multiplication

Project $(\widehat{x}, \widehat{y}, \widehat{z})$ to "normalized" image coordinates

$$x'_n = \widehat{x}/\widehat{z}$$
$$y'_n = \widehat{y}/\widehat{z}$$

Apply radial distortion

$$r^2 = {x'_n}^2 + {y'_n}^2$$
$$x'_d = x'_n(1 + \kappa_1 r^2 + \kappa_2 r^4)$$
$$y'_d = y'_n(1 + \kappa_1 r^2 + \kappa_2 r^4)$$

Apply focal length translate image center

$$x' = f x'_d + x_c$$
$$y' = f y'_d + y_c$$

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